Anisotropic generalization of Stinchcombe's solution for conductivity of random resistor network on a Bethe lattice

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Our study is based on the work of Stinchcombe [J. Phys. C **7**, 179 (1974)] and is devoted to the calculations of average conductivity of random resistor networks placed on a binary Bethe lattice. The binary structure of the Bethe lattice is assumed to represent the normal directions of the regular lattice. We calculate the anisotropic conductivity as an expansion in powers of inverse coordination number of the Bethe lattice. The expansion terms remained deliver an accurate approximation of the conductivity at resistor concentrations above the percolation threshold. This was confirmed by means of a Monte Carlo simulation on the regular 2D and 3D lattices.

I. INTRODUCTION

The random percolation theory due to Broadbent and Hammersley [1] is too simple to explain the great variety of percolation phenomena. One confronts complexity of real systems with both correlations and anisotropy playing important role. The motivation for our study is to understand better the nature of the anisotropy in electrical conductivity of percolating systems. This is approached by means of the random resistor network model (RRN) [2]. To our knowledge the first analytic solution for conductivity of anisotropic RRN on the regular square (2D) and cubic (3D) lattice was obtained by Bernaskoni [3]. Another useful result is that the resistor networks can be associated with the networks of saddle points in the conductivity profile of high-contrast systems as proved in [4]. Currently, anisotropic RRN models are studied in geophysics, which could possibly help in prediction of earthquakes [5, 6]. Besides conductivity, RRN has been used to predict magnetic properties of materials [7] and even to estimate sample destruction under critical mechanical stress [8]. The latter research provides a possible link to the science of earthquakes too [9].

Present study is based on RRN model and uses the exact solution on the isotropic Bethe lattice obtained by Stinchcombe [10]. This solution was demonstrated to describe the macroscopic conductivity curve for RRN on the regular 3D lattice calculated with a Monte Carlo (MC) simulation [11]. In addition, this solution can be used in connection with several important problems in solid-state physics [12]. We provide a generalization of the Stinchcombe's calculation to the case of a binary Bethe lattice, see Fig 1. Besides absence of closed loops, this structure has a special feature of being anisotropic at each node. Specifically, there are n_α bonds of α kind and n_β bonds of β kind connected at each branching point. At the same time, their total sum at a node is equal to a constant number z refereed to as the coordination number of the lattice. We would like to make a distinction between the finite Bethe lattice known also as the Caley tree and the infinite lattice with the surface sites neglected. The latter case is considered in the present contribution.

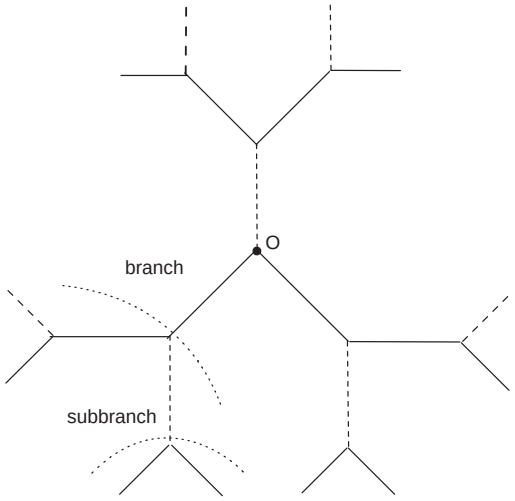

FIG. 1: Anisotropic Bethe lattice of coordination number $z = 3$ with two kinds of bonds, α and β, depicted by solid and dashed lines. The center O, referred to as origin, is where n_α and n_β branches are connected by their root bonds of α and β kind, respectively. $z = n_\alpha + n_\beta$. Each branch is made of $z - 1$ subbranches connected together by their root bonds.

The outline of the paper is as follows: in section II we present the model and mathematical formulation of the problem, in section III we present the main results, in section IV we describe our verification of the theory to MC simulations, in section V we provide a discussion and conclusions. Appendices A-C provide the details of our computation.

II. MODEL

Unlike several previous anisotropic percolation theories [13–17] based on the model of different probabilities of filling for two types of lattice bonds, we consider the distribution of resistors isotropic. However we make the local conductivities of the network elements on the binary

Bethe lattice the 'direction'-dependent, i.e. equal to σ_α or σ_β for α or β occupied bonds, respectively. Thus, the resistors are the occupied bonds of the lattice. The lattice itself is considered non-conductive. Mathematically, the local conductivity distribution function is written as follows:

$$g_\alpha(\sigma) = p\,\delta(\sigma - \sigma_\alpha) + (1-p)\delta(\sigma), \tag{1}$$

where p is the bond occupation probability, common for the bonds of both types. Given p and the conductivities of network elements, σ_α and σ_β, we compute the average conductivity of the network connected to a constant potential source at the origin and grounded at infinity. The question how to perform configurational averages turns out a difficult one.

A starting point of the present development is the observation that the percolation threshold is given by the usual equation [18]:

$$p_c = 1/(z-1), \tag{2}$$

as its derivation does not require considerations of conductivity as such. This is the consequence of the occupation probability common for the bonds of the both kinds. The isotropic percolation threshold of sticks of large length is expected from the Monte Carlo simulations [19] and observed experimentally in polymer/carbon-nanotube (CNT) composites [20].

Next one defines the probability distribution functions, $\phi_\alpha(b)$ and $\phi_\beta(b)$, of the branch conductivities b, such that

$$\int_0^\infty \phi_\alpha(b)db = 1. \tag{3}$$

Those functions measure the contribution from averaging over ensemble sampled by resistor permutations, so that the average branch conductivity is given by

$$\overline{b_\alpha} = \int_0^\infty b\phi_\alpha(b)db \tag{4}$$

Here, the symmetry $\alpha \leftrightarrow \beta$ holds for all quantities. Note that we specified in (3) and (4) only the components α for the sake of brevity. The second equation is obtained readily using $\alpha \leftrightarrow \beta$ interchange. This convention is followed everywhere in the text.

Therefore, the average conductivity of a part of the tree consisting of n_α branches connected at the origin in parallel is

$$\overline{\sigma_\alpha} = n_\alpha\,\overline{b_\alpha}. \tag{5}$$

As an example, one can take $n_\alpha = n$, and $n_\beta = z - n$, which leads to $n-1$ and $z-n$ of α- and β-subbranches, respectively, for the α branch (see Fig. 1). In order to compute $\phi_\alpha(b)$ and $\phi_\beta(b)$ we use the algorithm of Ref. [10] modified to account for the lattice binary structure, detailed calculation is placed to the Appendices A-C.

III. RESULTS

The analytical solutions have been obtained for the two cases: (I) for the case of infinitely large coordination number, $z \to \infty$ and (II) near the percolation threshold, $p \approx p_c$. In both cases the solution is represented in the form of a Taylor expansion in terms of the small parameters, $p_c = (z-1)^{-1}$ and $\epsilon = (p - p_c)/p_c$, respectively.

In the first case we obtain (see Appendix B for details)

$$\overline{b_{\alpha(\mathrm{I})}} = -\sigma_\alpha\left(-\frac{p\sigma_\beta\Delta}{\sigma_\alpha p_c + \sigma_\beta\Delta} + \frac{\Delta}{p}\sum_{k=2}^{\infty} G^{(k)}\right), \tag{6}$$

where

$$\begin{aligned}
G^{(2)} &= \frac{p_c^2}{p^2}\,\Delta\, s,\\
G^{(3)} &= \frac{p_c^3}{p^5}\,\Delta^2 s[p(2p-1) + 3\Delta s],\\
G^{(4)} &= \frac{p_c^4}{p^6}\,\Delta^2 s\,\big[3s^2\Delta - 2sp\\
&\quad +\Delta(1 - 3p + 3p^2)\\
&\quad +10\Delta^2 s\left(\frac{2p-1}{p}\right) + 15\Delta^3 s^2\frac{1}{p^2}\Big],\\
G^{(k)} &= \mathrm{O}(p_c^k),\\
\Delta &= p - p_c, \quad \text{and} \quad s = 1 - p.
\end{aligned} \tag{7}$$

This equation gives the average conductivity of a branch starting from the α-bond connected to a potential difference between the node at its root and the nodes at infinity. The first term is the conductivity of the infinitely branched Bethe lattice, while the summation over G-s represents the corrections up to and including $(z-1)^{-4}$ order. Substituting $\sigma_\alpha = \sigma_\beta$ into eq. (6) leads to the formula that differs from that in Ref. [10] by the factor Δ/p before the summation. We carefully verified this and will provide the comparison of our modified and the original result to the Monte Carlo (MC) data available in literature, see the next Section.

We now calculate the critical exponents and the anisotropy near the percolation threshold, $p \approx p_c$. The details of the calculation are shown in the Appendix C. Close the critical point the integer numbers n_β and n_α are set by

$$n_\alpha\sigma_\beta = n_\beta\sigma_\alpha, \tag{8}$$

received from the symmetry considerations, eq. (C.14). Qualitatively, this can be explained as follows: Bethe lattice, with its origin O representing a point inside the sample, has the branching topology of the infinite cluster. Suppose the system is just above p_c. The infinite cluster should form first in the direction where the resistance to current is minimal. For the case when the occupation probability p is the same in all directions, the direction dependent percolation probability can only be achieved

if the fraction of bonds of one kind is larger than another. Indeed, the isotropic percolation probability $P = 1 - R^z$, where $R < 1$ is the probability to have the finite cluster [18], can be generalized to the anisotropic one, $P_\alpha = 1 - R^{n_\alpha}$, which gives $P_\alpha > P_\beta$ if $n_\alpha > n_\beta$. This leads to the following conditions:

$$n_\alpha = \frac{z\sigma_\alpha}{\sigma_\alpha + \sigma_\beta}, \quad n_\beta = z - n_\alpha. \tag{9}$$

Thus, n_α and n_β are fixed by the local conductivities.

Returning for a moment to the previous case, we note that the Bethe lattice topology should be intact on change of p. Thus, the condition (9) has to be applied above the critical point to obtain $\overline{\sigma_\alpha}_{(\mathrm{I})}$ from eqs. (5) and (6):

$$\overline{\sigma_\alpha}_{(\mathrm{I})} = \frac{z\sigma_\alpha}{\sigma_\alpha + \sigma_\beta}\overline{b_\alpha}_{(\mathrm{I})} \tag{10}$$

In the critical region, we investigate the anisotropy ratio of the network conductivities and relate this to the experimental quantity $\overline{\sigma_{||}}/\overline{\sigma_\perp}$, where $\overline{\sigma}_{||,\perp}$ are the bulk conductivities parallel and normal to the direction of an applied voltage. According to Skal and Shklovskii [22],

$$\overline{\sigma_{||}}/\overline{\sigma_\perp} - 1 \simeq (p - p_c)^{\lambda(d)}, \tag{11}$$

where $\lambda(d)$ is a critical exponent determined by d - the dimensionality of a problem. Sarychev and Vinogradov[23] using the renormalization group theory and computer simulations found that $\lambda(2) = 0.9 \pm 0.1$ and $\lambda(3) = 0.3 \pm 0.1$ for 2D and 3D, respectively. Carmona and Amarti[24] deduced from experimental data for short carbon fiber reinforced polymers that $\lambda(3) \approx 0.4$. The details of our computation are placed in the Appendix C. Our final result (C.21), written in a more concise form, is given by

$$\overline{\sigma_\alpha}_{(\mathrm{II})} = 0.762\frac{z}{z-2}\frac{2\sigma_\alpha\sigma_\beta}{\sigma_\alpha + \sigma_\beta}\epsilon^2 + O(\epsilon^3), \tag{12}$$

where $\epsilon = (p - p_c)/p_c$. The discrepancy of the critical exponent of 2 with the value of 3 in the case of infinite dimensions examined by de Gennes [25], we assign to the category of unresolved problems especially because de Gennes uses the concept of surface that is never considered in Bethe lattice theories. It is known also that the critical exponents σ and τ in the classical Flory-Stockmayer theory of the sol-gel transition formulated on Bethe lattice are close to those at 3 dimensions [26]. In the conductivity field, the critical exponent 2 is assigned as the standard one for 3D systems [18, 27].

We also calculated the average anisotropy ratio near the percolation threshold (C.23):

$$\overline{\sigma_\alpha}_{(\mathrm{II})}/\overline{\sigma_\beta}_{(\mathrm{II})} - 1 = \frac{z-1}{z}\frac{\sigma_\alpha^2 - \sigma_\beta^2}{\sigma_\alpha\sigma_\beta}\epsilon + \mathrm{O}(\epsilon^2). \tag{13}$$

This result is analogous to (11) for the case when α and β are associated with the parallel and the perpendicular components, respectively. Thus, we receive the critical exponent, $\lambda = 1$, which is consistent with the exponent obtained in Ref. [28] by an analogous method.

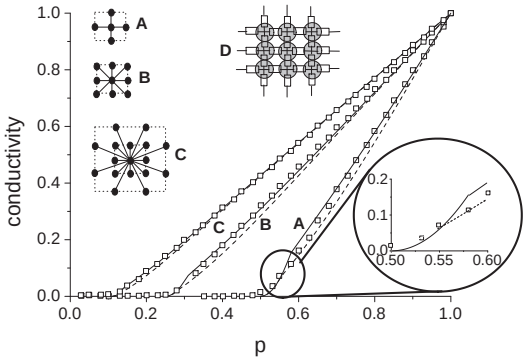

FIG. 2: Comparison of the near-mean-field formula of Ref. [10] (dashed curves) and $\overline{\sigma}_{(\mathrm{I}),(\mathrm{II})}$ given by eqs. (10),(12) with $\sigma_\alpha = \sigma_\beta$ (continuous curves) with the MC simulation on 2D lattice [21]. The inset D shows a resistor network on 2D lattice and the dual network of circles with four nearest neighbors. The insets A-C show the networks of circles with multiple nearest neighbors. Bethe lattice coordination number used is $z = 3, 5, 10$ for the number neighbors on 2D lattice equal to $4, 8, 16$, respectively.

IV. COMPARISON WITH MONTE CARLO SIMULATIONS

In the work of Strelniker [21] the MC data on a regular 2D lattice, for a case when the number of neighbors is greater than four, are shown. The insets A-D in the Fig. 2 demonstrate the model used by Strelniker to reduce the resistor network problem on the square lattice to the problem of multiple nearest neighbors. The dashed curves represent the original formula, the continuous curves show eqs. (10) and (12) with $\sigma_\alpha = \sigma_\beta$ and $z = 3, 5, 10$ for the cases A,B,C respectively. We have chosen these values to match the data extracted from the graph in Ref. [21]. Since the only available limits investigated are $z \to \infty$ and $p \approx p_c$ and there is no connection between them, we just glued these two solutions at their intersection point as seen in the blow up (Fig. 2). It is clear that our correction provides a better fit well above the percolation threshold, while the original formula seems to be smoother close to p_c.

The analytical results (10) and (12) need to be verified by the Monte Carlo computer modeling on the regular square (2D) and cubic (3D) lattice. For this purpose, we performed the simulation of resistor networks with the resistor values different for horizontal and vertical lattice directions. In the simulation box, two opposite boundaries, see 2D scheme in the insets of Fig. 3, are associated with the electrodes that are kept at some fixed potentials V = const and 0, whereas the other surfaces of the box are taken with the periodic boundary conditions. We used the square lattice of size 50×50 and the cubic lattice $18 \times 18 \times 18$. Our computer program solves the Kirchhoff equations at each lattice node using the

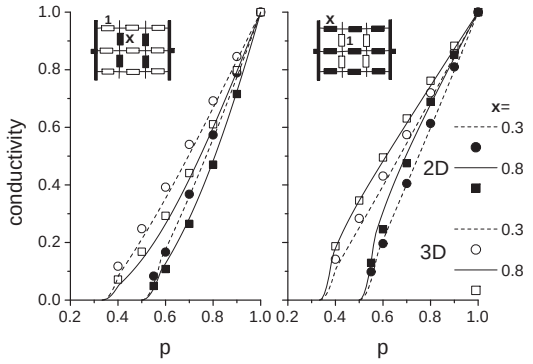

FIG. 3: Comparison between the MC simulation on 2D,3D square lattices and the Bethe lattice theory. The theory predictions with $z = 3$ and 4 for 2D and 3D, respectively, are shown by the curves. The inset shows a resistor network on fully occupied regular lattice ($p = 1$). Resistors are shown by two colors to elucidate their orientation-dependent conductivities. Two electrodes are connected to an external source of constant voltage. Periodic boundary conditions are set in the perpendicular direction.

iteration-relaxation method described in [2]: 1000 random configurations are generated for each p and the α parameter from Ref. [2] to get the average conductivity. The advantage of the iteration-relaxation method in comparison with the alternative transfer-matrix method [29] is in relative simplicity of the former. However, the lack of the iteration-relaxation method is in progressive computational delay upon entering the threshold proximity.

Present analytical results and the simulation data are compared in Fig. 3. Continuous curves correspond to the theory, filled and open squares (circles) represent MC data for the average conductivity of occupied bonds of type α (β) on the regular 2D and 3D lattices, respectively. For the comparison with these MC data, the Bethe lattice of $z = 3$ and 4 was used for 2D and 3D, respectively. Thus, in view of quite good conformity of the theory for $z = 3, 4$ and MC data, the fact of fast convergence of the $(z-1)^{-1}$ expansion is obvious. Furthermore, we find that the factor Δ/p in (6) is crucial for the anisotropic case, especially for the averaged component corresponding the preferential conductivity direction.

V. DISCUSSION AND CONCLUSIONS

We proposed the model of the resistor network that has a property of anisotropy in a sense that the conductivities of the resistors differ with respect to the lattice bond type. We solve the problem in the framework of the anisotropic Bethe lattice approximation. The mathematical problem was formulated in terms of a nonlinear integral equation, which was solved asymptotically using series expansions in two limiting cases: near the percolation threshold and near the mean-field limit of $z \to \infty$.

As noticed in [5], significant anisotropies observed in geophysics at the macroscale could be explained by the formation of fractal structures in a microscale. Here, the macroscopic observable anisotropy is the property of entire network and the local (intrinsic) anisotropy is associated with the anisotropy in conductivity at a branching point of the Bethe lattice. The former is defined as $\overline{a} = \overline{\sigma_\alpha}/\overline{\sigma_\beta}$, whereas the latter is essentially the ratio $a = \sigma_\alpha/\sigma_\beta$ being the *only* parameter entering eqs. (6,7,10). We find that the present theory is capable of producing the strong global anisotropy, $\overline{a}$, at small local anisotropy, a, in the case when z is large and $p \gg p_c$. Indeed, in this limit the conductivity is given by the mean-field formula: $\overline{\sigma_\alpha}_{(\mathrm{I})} \approx zp\sigma_\alpha^2/(\sigma_\alpha+\sigma_\beta)$, which yields $\overline{a}_{(\mathrm{I})} \approx a^2$. Interestingly, that previous theories based on anisotropic occupation probability [14, 17] predicted the opposite: at strong local anisotropy - weak global one.

Returning to the insets A-C of Fig. 2 we observe that the number of the special directions corresponding to the vertical or horizontal remains finite as $z \to \infty$. This is in conformity with the assumption $z \gg n$ used in the Appendix B, but one has to keep in mind that this is only a qualitative explanation. Bethe lattice cannot be embedded in a finite dimensional space, which suggests that, generally speaking, one cannot give a simple geometrical picture relating the local anisotropy of the binary Bethe lattice to some normal space coordinates.

VI. ACKNOWLEDGMENTS

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APPENDIX A: ANISOTROPIC BETHE LATTICE THEORY [10, 30]

For a branch and its $z-1$ next generation subbranches (see Fig. 1) a set of conductivities is defined:

$$\{b_\alpha^{(i)}\} = b_\alpha^{(0)}, b_\alpha^{(1)}, ..., b_\alpha^{(n_\alpha-1)}, b_\beta^{(n_\alpha)}, ..., b_\beta^{(z-1)},$$

which are zero or finite according as the corresponding root bonds are empty or occupied, where the index $i = 0$ is reserved for the branch origin. These subbranches are connected in parallel, so that the conductivities $b_\alpha^{(0)}$ are given by

$$b_\alpha^{(0)} = \sum_{i=1}^{n_\alpha-1} \frac{\sigma_\alpha b_\alpha^{(i)}}{\sigma_\alpha + b_\alpha^{(i)}} + \sum_{i=n_\alpha}^{z-1} \frac{\sigma_\beta b_\beta^{(i)}}{\sigma_\beta + b_\beta^{(i)}} \qquad \text{(A.1)}$$

Both $b_\alpha^{(i)}$ ($b_\beta^{(i)}$) and σ_α (σ_β), which are the branch and bond conductivities, respectively, are random variables distributed with some probability density functions. In order to determine the branch distribution functions $\phi_\alpha(b^{(0)})$ defined in (3) and (4) we average over various resistor configurations on the lattice using the distribution functions $\phi_\alpha(b^{(i)})$ and $g_\alpha(\sigma^{(i)})$ defined for the conductivities of subbranches and individual bonds, respectively, so that

$$g_\alpha(\sigma) = p\,\delta(\sigma-\sigma_\alpha) + (1-p)\delta(\sigma). \tag{A.2}$$

Being more specific we determine $\phi_\alpha(b)$ [note that the superscript (0) is suppressed for brevity] by performing an asymptotic analysis of

$$\begin{aligned}\phi_\alpha(b) \;=\; & \prod_{i=1}^{n_\alpha-1}\left(\int_0^\infty d\sigma^{(i)}\, g_\alpha(\sigma^{(i)})\int_0^\infty db^{(i)}\,\phi_\alpha(b^{(i)})\right)\times \\ & \prod_{i=n_\alpha}^{z-1}\left(\int_0^\infty d\sigma^{(i)}\, g_\beta(\sigma^{(i)})\int_0^\infty db^{(i)}\,\phi_\beta(b^{(i)})\right)\times \\ & \delta\left(b-b_\alpha^{(0)}\right),\end{aligned} \tag{A.3}$$

Since $\phi_\alpha(b)$ is actually a series of delta functions, it is convenient to introduce the Laplace transform of $\phi_\alpha(b)$, generally known as the moment-generating function

$$B_\alpha(q) \equiv \int_0^\infty e^{-qb}\phi_\alpha(b)db. \tag{A.4}$$

In the present study, this quantity is named the branch generating function. Equation (A.4) combined together with (4) leads to

$$\bar{b}_\alpha = -B_\alpha^{'}(0), \tag{A.5}$$

which means the average branch conductivity is just the negative first derivative of $B_\alpha(q)$ evaluated at $q=0$. Taking the Laplace transform of eq.(A.3) it can be shown that

$$B_\alpha(q) = C_\alpha(q)^{n_\alpha-1}C_\beta(q)^{n_\beta}, \tag{A.6}$$

where

$$\begin{aligned}C_\alpha(q) = & \\ & \int_0^\infty d\sigma\, g_\alpha(\sigma)\int_0^\infty db\,\phi_\alpha(b)\exp\left(-\frac{q\sigma b}{\sigma+b}\right).\end{aligned} \tag{A.7}$$

Therefore, on account of (A.6), $C_\alpha(q)$ and $C_\beta(q)$ can be named the subbranch generating functions and, in analogy with (A.5), one can define the subbranch average conductivity as

$$\bar{b}_\alpha^{(i)} = -C_\alpha^{'}(0). \tag{A.8}$$

Since $\phi_\alpha(b)$ and $g_\alpha(\sigma)$ are the probability densities normalized to unity, see (3) and (A.2), respectively, the boundary condition for $C_\alpha(q)$ at $q=0$ is

$$C_\alpha(0) = 1. \tag{A.9}$$

The other boundary condition at $q=\infty$ is identified as the probability to have the finite cluster, R, since the main contribution to the integral (A.7) comes from the neighborhood of $b=0$ or $\sigma=0$:

$$C_\alpha(\infty) = R. \tag{A.10}$$

After some algebra [10] which involves an additional Laplace transform that introduces a new variable t, one obtains the integral equation

$$\begin{aligned}\int_0^\infty e^{-tq}C_\alpha(q)dq = & \int d\sigma g_\alpha(\sigma)(t+\sigma)^{-1}\times \\ & \left[1+\frac{\sigma^2}{t+\sigma}\int_0^\infty \exp\left(-\frac{q\sigma t}{\sigma+t}\right)B_\alpha(q)dq\right]\end{aligned} \tag{A.11}$$

which is the final exact result to be solved asymptotically.

APPENDIX B: NEAR-MEAN-FIELD EXPANSION

Consider the integrals on both sides of eq. (A.11). These integrals will be approximated for large t values using the Laplace method.[31] The method is based on the idea that the main contribution to the integrals comes from the neighborhood of $q=0$, which makes it possible to use the Taylor series expansion as follows

$$\begin{aligned}C_\alpha(q) &= e^{\ln\left[C_\alpha(0)+qC_\alpha^{'}(0)+\ldots\right]} \\ &= e^{qC_\alpha^{'}(0)}\left[1+\sum_{l=2}^\infty a_\alpha^{(l)}q^l\right], \quad \text{for} \quad q\ll 1.\end{aligned} \tag{B.1}$$

This defines $a_\alpha^{(l)}$ multiplies. Additionally, we have $b_\alpha^{(l)}$-coefficients by

$$C_\alpha(q)^m = e^{mqC_\alpha^{'}(0)}\sum_{l=0}^\infty b_\alpha^{(l)}q^l. \tag{B.2}$$

Substituting (B.2) into (A.11), one obtains

$$\begin{aligned}\int_0^\infty e^{-tq}C_\alpha(q)dq = & \int du g_\alpha(u) \\ & \left\{\frac{1}{t-\tau_\alpha(u)}+\frac{u^2}{(t+u)^2}\sum_{k=0}^\infty b_\alpha^{(k)}\frac{k!}{s_\alpha^{k+1}}\right\},\end{aligned} \tag{B.3}$$

where

$$s_\alpha \;=\; \frac{ut}{u+t} - (n_\alpha-1)C_\alpha^{'}(0) - n_\beta C_\beta^{'}(0) \tag{B.4}$$

$$\tau_\alpha(u) \;=\; \frac{u\left[(n_\alpha-1)C_\alpha^{'}(0)+n_\beta C_\beta^{'}(0)\right]}{u-(n_\alpha-1)C_\alpha^{'}(0)-n_\beta C_\beta^{'}(0)}. \tag{B.5}$$

The formula (B.3) represents an expansion in inverse powers of z which is seen from (B.4,B.5). Inversion of

the Laplace transform in (B.3) yields

$$C_\alpha(q) = \int du\, g_\alpha(u)\, \exp\left[q\tau_\alpha(u)\right] \times \left[1 + \sum_{k=2}^{\infty} b_\alpha^{(k)} k! \sum_{r=0}^{k-1} \frac{{}_{k-1}C_r}{(k-r)!} \times \frac{u^{2(k-r)} q^{k-r}}{\left[u - (n_\alpha - 1)C_\alpha'(0) - n_\beta C_\beta'(0)\right]^{2k-r}}\right],$$

where ${}_{k-1}C_r$ are the binomial coefficients. The equation right above is combined with (B.1), then one equates term by term the factors of the successive powers of q, and obtains

$$0 = \int du g_\alpha(u)$$

for $l = 0$ and

$$0 = \int du g_\alpha(u)[\tau_\alpha(u) - C_\alpha^{'}(0)] + \sum_{m=2}^{\infty} b_\alpha^{(m)} I_\alpha^{(m10)} \quad \text{(B.6)}$$

$$a_\alpha^{(k)} = a_\alpha^{(k)0} + \sum_{m=2}^{\infty} b_\alpha^{(m)} \times \sum_{s=0}^{\min\{m-1,\,k-1\}} I_\alpha^{(mks)},\ k \geq 2, \quad \text{(B.7)}$$

for $l = 1$, where

$$a_\alpha^{(k)0} = \int_0^\infty du\, g_\alpha(u) \frac{\left[\tau_\alpha(u) - C_\alpha^{'}(0)\right]^k}{k!} \quad \text{(B.8)}$$

and

$$I_\alpha^{(mks)} = \frac{m!\,{}_{m-1}C_s}{(s+1)!\,[k-(s+1)]!} \int du g_\alpha(u) \times \frac{u^{2(s+1)}[\tau_\alpha(u) - C_\alpha^{'}(0)]^{k-(s+1)}}{[u - (n_\alpha - 1)C_\alpha'(0) - n_\beta C_\beta'(0)]^{m+s+1}}. \quad \text{(B.9)}$$

The first term on the right-hand-side of (B.6) is the representation of the mean-field limit $z \to \infty$,

$$0 = \int du g_\alpha(u)[\tau_\alpha(u) - \overline{C}_\alpha^{'}(0)], \quad \text{(B.10)}$$

and the sum over m gives the corrections in inverse powers of z. Two equations, obtained by the interchange of α and β in (B.10), will be solved neglecting $n_\alpha = n$ as it is a constant negligibly small compared to $z - 1$. [Here, in order to keep up with the Stinchcome's results, we expand in powers of inverse $z - 1$ and not z, which is equivalent]. Solving (B.10) we get, as the first solution, the isotropic mean-field conductivity:

$$\overline{C}_\alpha^{'}(0)_{\rm iso} = -\sigma_\alpha(p - p_c). \quad \text{(B.11)}$$

Additionally, we obtain

$$\overline{C}_\alpha^{'}(0) = -\frac{\sigma_\alpha\sigma_\beta(p - p_c)p}{\sigma_\alpha p_c + \sigma_\beta(p - p_c)}, \quad \text{(B.12)}$$

which is the anisotropic solution of main interest for us.

We now move to some elaboration regarding the orders of correction contained in (B.6)-(B.9). One finds that $I_\alpha^{(210)} b_\alpha^{(2)}$ is of the order $(z-1)^{-2}$, since $I_\alpha^{(m10)}$ and $b_\alpha^{(m)}$ are of the orders $(z-1)^{-(m+1)}$ and $(z-1)^{m/2}$, respectively. Note that the correction to $a_\alpha^{(2)0}$ given by the first term of the sum in (B.7) affects $I_\alpha^{(210)} b_\alpha^{(2)}$ by $(z-1)^{-4}$ order. Thus, to have the final result up to and including $\mathrm{O}([z-1]^{-4})$, the first term in the sum given by (B.7) should be taken into account, but only $a_\alpha^{(m)0}$ can be used for $m > 2$. In addition, when approximating $a_\alpha^{(m)}$ and $I_\alpha^{(m10)}$ with $m > 2$, we use the replacement $C_\alpha^{'}(0) = \overline{C}_\alpha^{'}(0)$. This is perfectly acceptable if $m > 2$, since any correction to this would be of $(z-1)^{-2}$ order, and hence would contribute to $a_\alpha^{(m)0}$ as $(z-1)^{-2m}$. To compute the error introduced by this substitution for $m = 2$, we expand $T_\alpha = I_\alpha^{(210)} a_\alpha^{(2)0}$ near $\bar{T}_\alpha = \overline{I}_\alpha^{(210)} \overline{a}_\alpha^{(2)0}$:

$$T_\alpha = \bar{T}_\alpha + (D^\alpha T_\alpha)\Delta C_\alpha + (D^\beta T_\alpha)\Delta C_\beta, \quad \text{(B.13)}$$

where $\Delta C_\alpha = C_\alpha^{'}(0) - \overline{C}_\alpha^{'}(0)$, $D^\beta T_\alpha = [\partial T_\alpha / \partial C_\beta^{'}(0)]_{\overline{C}_\beta^{'}(0)}$. Additionally, from (B.6) with the term $m = 2$ only, one has

$$-\bar{T}_\alpha = (D^\alpha A_\alpha)\Delta C_\alpha + (D^\beta A_\beta)\Delta C_\beta \quad \text{(B.14)}$$

where $D^\beta A_\alpha = [\partial \int du g_\alpha(u)[\tau_\alpha - C_\alpha^{'}(0)]/\partial C_\beta^{'}(0)]_{\overline{C}_\beta^{'}(0)}$. The computation of the coefficients yields

$$\begin{aligned} D^\alpha T_\alpha &= -1 + \mathrm{O}([z-1]^{-1}), \\ D^\beta T_\alpha &= 0, \\ D^\alpha A_\alpha &= 0, \\ D^\beta A_\alpha &= (z-1)\overline{I}_\alpha^{(310)} \overline{b}_\alpha^{(2)0}. \end{aligned} \quad \text{(B.15)}$$

Equations (B.13)-(B.15) combined together give

$$T_\alpha = \bar{T}_\alpha \left(1 + (z-1)\overline{I}_\alpha^{(310)} \overline{b}_\alpha^{(2)0} \frac{\bar{T}_\beta}{\bar{T}_\alpha}\right) = \bar{T}_\alpha + \mathrm{O}([z-1]^{-4}). \quad \text{(B.16)}$$

Substituting this into (B.6), one obtains the expression that contains all corrections up to $(z-1)^{-4}$ order:

$$0 = \int g_\alpha(u) \left[\tau_\alpha(u) - C_\alpha^{'}(0)\right] + (z-1) a_\alpha^{(2)0} I_\alpha^{(210)} \times \left\{1 + (z-1) I_\alpha^{(220)} + (z-1)^2 a_\alpha^{(2)0} I_\alpha^{(310)} \frac{I_\beta^{(210)}}{I_\alpha^{(210)}}\right\} + I_\alpha^{(310)} b_\alpha^{(3)0} + I_\alpha^{(410)} b_\alpha^{(4)0} + I_\alpha^{(510)} b_\alpha^{(5)0} + I_\alpha^{(610)} b_\alpha^{(6)0}. \quad \text{(B.17)}$$

Direct computation of the eqs. (B.8) and (B.9) yields

$$\overline{a}_\alpha^{(m)0} = \frac{1}{m!}(-1)^m \overline{C}_\alpha^{'}(0)^m (1-p) \times \left[1+(-1)^m \left(\frac{1-p}{p}\right)^{m-1}\right], \quad \text{(B.18)}$$

$$\overline{I}_\alpha^{(m10)} = \frac{m!\left[\overline{C}_\alpha^{'}(0)+\sigma_\alpha\, p\right]^{m+1}}{\sigma_\alpha^{m-1}\, p^{\,m}} \quad \text{(B.19)}$$

$$\overline{I}_\alpha^{(220)} = \frac{2(1-p)\overline{C}_\alpha^{'}(0)\left[\overline{C}_\alpha^{'}(0)+\sigma_\alpha\, p\right]^3}{\sigma_\alpha\, p^{\,2}} \quad \text{(B.20)}$$

Finally, substituting (B.18)-(B.20) into (B.17) and using (A.5), we get the anisotropic conductivity in the form of a series expansion in successive powers of the inverse coordination number:

$$\overline{b_{\alpha(\mathrm{I})}} = -\sigma_\alpha \left(-\frac{p\sigma_\beta \Delta}{\sigma_\alpha p_c + \sigma_\beta \Delta} + \frac{\Delta}{p}\sum_{n=2}^{\infty} G^{(n)}\right), \quad \text{(B.21)}$$

where

$$\begin{aligned} G^{(2)} &= \frac{p_c^2}{p^2}\,\Delta\, s, \\ G^{(3)} &= \frac{p_c^3}{p^5}\,\Delta^2 s[p(2p-1)+3\Delta s], \\ G^{(4)} &= \frac{p_c^4}{p^6}\,\Delta^2 s\left[3s^2\Delta - 2sp + \Delta(1-3p+3p^2)\right. \\ &\quad \left. +10\Delta^2 s\left(\frac{2p-1}{p}\right) + 15\Delta^3 s^2 \frac{1}{p^2}\right], \\ G^{(n)} &= \mathrm{O}(p_c^n), \\ \Delta &= p - p_c, \quad \text{and} \quad s = 1-p. \end{aligned} \quad \text{(B.22)}$$

The expansion (B.22) coincides with the result of Ref. [10] except that factor 5 in front of $\Delta^2 s\left(\frac{2p-1}{p}\right)$ for $G_\alpha^{(4)}$ needs to be replaced by 10 according to us.

APPENDIX C: INVESTIGATION OF CRITICAL INDICES

In this appendix we investigate the critical exponents by means of an asymptotic analysis of the integral equation (A.11) for p approaching p_c from above introducing a small parameter

$$\epsilon = \frac{p-p_c}{p_c}. \quad \text{(C.1)}$$

On the one hand, it is known [18] that the percolation probability P, defined on the Bethe lattice as $P = 1-R^z$, can be expanded near the percolation threshold in series:

$$P(\epsilon) = B\epsilon + C\epsilon^2 + \mathrm{O}(\epsilon^3),$$

where B and C are constants and hence

$$R = 1 - \delta^{(1)}\epsilon - \delta^{(2)}\epsilon^2 + \mathrm{O}(\epsilon^3). \quad \text{(C.2)}$$

Here,

$$\delta^{(1)} = 2/(z-2), \quad \text{(C.3)}$$

while the numerical value of $\delta^{(2)}$ has no significance for us, as shown in the analysis set forth below.

On the other hand, the anisotropic conductivity expansion in terms of ϵ has not previously been addressed. Motivated by the analysis of Ref. [10], we propose a trial solution to (A.11) of the form

$$C_\alpha(q) = R + \epsilon C_\alpha^{(1)}(q) + \epsilon^2 C_\alpha^{(2)}(q), \quad \text{(C.4)}$$

where $C_\alpha^{(1,2)}(q)$ are slowly varying functions of q described by the scaling relations

$$C_\alpha^{(1,2)}(q) = f_\alpha^{(1,2)}(c_\alpha \epsilon q), \quad \text{(C.5)}$$

with c_α being a constant to be determined later.

Now, the variables $s = t/(c_\alpha\epsilon)$ and $y = c_\alpha \epsilon q$ are defined. Substituting (C.2) and (C.4) into the left-hand-side of (A.11), multiplying both sides by t and expressing the integrals in terms of the new variables, one finds

$$\int_0^\infty e^{-sy}[C_\alpha(y)-1]dy = p_c \frac{\sigma_\alpha(1+\epsilon)}{(\sigma_\alpha + c_\alpha\epsilon s)^2} \times \int_0^\infty \exp\left(-\frac{sy\sigma_\alpha}{\sigma_\alpha + c_\alpha \epsilon s}\right)\left\{C_\alpha^{n_\alpha - 1} C_\beta^{n_\beta} - 1\right\} dy, \quad \text{(C.6)}$$

where $C_\alpha(y)$ are given by (C.4) and (C.5). A set of equations is obtained equating the ϵ-expansion coefficients of the same order in the left- and right-hand-side of the integral equation (C.6). Firstly, equating the terms linear in ϵ we obtain

$$f_\alpha^{(1)}(y) = p_c[(n_\alpha - 1)f_\alpha^{(1)}(y) + n_\beta f_\beta^{(1)}(y)]. \quad \text{(C.7)}$$

As a result, the first correction is isotropic,

$$f_\alpha^{(1)}(y) = f_\beta^{(1)}(y) \equiv f^{(1)}(y). \quad \text{(C.8)}$$

Secondly, equating the terms proportional to ϵ^2 and using (C.8) yields

$$\int_0^\infty e^{-sy} f_\alpha^{(2)}(y)dy = \int_0^\infty dy e^{-sy} \times \left\{\frac{1}{\delta^{(1)}}(f^{(1)} - \delta^{(1)})^2 + (f^{(1)} - \delta^{(1)}) \times \left[1 + \frac{c_\alpha}{\sigma_\alpha}(-2s + s^2 y)\right] + p_c[(n_\alpha - 1)f_\alpha^{(2)} + n_\beta f_\beta^{(2)}]\right\}. \quad \text{(C.9)}$$

Isotropic solution: When substituting $c_\alpha = \sigma_\alpha$, we recover the isotropic solution $f_\alpha^{(2)} = f_\beta^{(2)} = f^{(2)}$ satisfying

$$\int_0^\infty e^{-sy} f^{(2)}(y)dy = \int_0^\infty dy e^{-sy} \times \left\{\frac{1}{\delta^{(1)}}(f^{(1)} - \delta^{(1)})^2 + (f^{(1)} - \delta^{(1)})\left[1 - 2s + s^2 y\right] + p_c[(n_\alpha - 1)f^{(2)}(y) + n_\beta f^{(2)}(y)]\right\}.$$

and

$$0 = \int_0^\infty dy e^{-sy}\left\{\frac{1}{\delta^{(1)}}(f^{(1)}-\delta^{(1)})^2 + (f^{(1)}-\delta^{(1)})\left[1-2s+s^2y\right]\right\} \quad \text{(C.10)}$$

This gives a simple solution:

$$f^{(1)}(y)_{\rm iso} = \delta^{(1)}\xi(y), \quad \text{(C.11)}$$

where ξ is determined by solving numerically the differential equation of the second order,

$$y\xi'' = \xi(1-\xi), \quad \xi(0)=1, \quad \xi(\infty)=0, \quad \text{(C.12)}$$

which gives $\xi'(0) = -0.762$.

Anisotropic solution: Here, we try $c_\alpha \neq \sigma_\alpha$ in (C.9) to obtain another solution. To simplify (C.9), we combine it with (C.10), which yields

$$\int_0^\infty e^{-sy} f_\alpha^{(2)}(y)dy = \int_0^\infty e^{-sy}\left\{(f^{(1)}-\delta^{(1)})\left[\frac{c_\alpha}{\sigma_\alpha}-1\right](-2s+s^2y) + p_c[(n_\alpha-1)f_\alpha^{(2)}(y) + n_\beta f_\beta^{(2)}(y)]\right\}dy.$$

From this equation follows that

$$f_\alpha^{(2)} - p_c[(n_\alpha-1)f_\alpha^{(2)} + n_\beta f_\beta^{(2)}] = \frac{c_\alpha-\sigma_\alpha}{\sigma_\alpha}\delta^{(1)}x\xi''(x), \quad \text{(C.13)}$$

where x is an arbitrary variable. The quantities $f_\alpha^{(2)}$(x) and $f_\beta^{(2)}$(x) have to be symmetric with respect to $\alpha \leftrightarrow \beta$ interchange. This condition is satisfied only in the two following cases:

$$n_\alpha\sigma_\beta = n_\beta\sigma_\alpha, \quad \text{(C.14)}$$

$$c_\alpha = \sigma_\beta, \quad \text{(C.15)}$$

or

$$n_\alpha = n_\beta, \quad \text{(C.16)}$$

$$c_\alpha = 2\sigma_\alpha\sigma_\beta/(\sigma_\alpha+\sigma_\beta). \quad \text{(C.17)}$$

The second set of conditions, eqs. (C.16) and (C.17), is unappropriate here due to the condition $z >> n$ utilized in the Appendix B. Hence, from (C.14) we have

$$n_\alpha = z\sigma_\alpha/(\sigma_\alpha+\sigma_\beta), \quad n_\beta = z - n_\alpha. \quad \text{(C.18)}$$

Substitution of (C.18) into (C.13) yields

$$f_\alpha^{(2)}(x) - f_\beta^{(2)}(x) = -\delta^{(1)}x\xi''(x)\frac{z-1}{z}\frac{\sigma_\alpha^2-\sigma_\beta^2}{\sigma_\alpha\sigma_\beta}. \quad \text{(C.19)}$$

The latter is symmetric with respect to α and β interchange. Finally, combining (C.4,C.5,C.3,C.11,C.15), we write the anisotropic solution:

$$C_\alpha(q) = R + \frac{2}{z-2}\xi(\sigma_\beta\epsilon q)\epsilon + f_\alpha^{(2)}(\sigma_\beta\epsilon q)\epsilon^2. \quad \text{(C.20)}$$

Then, using (A.5), (5) and (C.18), we obtain the average conductivity up to and including ϵ^3 order:

$$\overline{\sigma_\alpha}_{\rm (II)} = \frac{z\sigma_\alpha}{\sigma_\alpha+\sigma_\beta}\left\{0.762\frac{2}{z-2}\sigma_\beta\epsilon^2 - \sigma_\beta f_\alpha^{(2)'}(q)\epsilon^3\right\} + C\epsilon^3, \quad \text{(C.21)}$$

where C is a constant which could be determined by equating the terms proportional to ϵ^3 in the expansion of eq. (C.6). As a result the first term in the expansion (C.21), proportional to ϵ^2, is the symmetric one. In addition, we can calculate the difference in the derivatives of $f_\alpha^{(2)}$ and $f_\beta^{(2)}$ using eqs. (C.19,C.12), which yields

$$\overline{\sigma_\alpha}_{\rm (II)} - \overline{\sigma_\beta}_{\rm (II)} = 0.762\frac{2(z-1)}{z-2}(\sigma_\alpha-\sigma_\beta)\epsilon^3. \quad \text{(C.22)}$$

Therefore, we have

$$\frac{\overline{\sigma_\alpha}_{\rm (II)} - \overline{\sigma_\beta}_{\rm (II)}}{\overline{\sigma_\beta}_{\rm (II)}} = \frac{z-1}{z}\frac{\sigma_\alpha^2-\sigma_\beta^2}{\sigma_\alpha\sigma_\beta}\epsilon. \quad \text{(C.23)}$$

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